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Quantization by non-Abelian promeasures*

C J S Clarke

Faculty of Mathematical Studies, University of Southampton, Southampton SO9 5NH, UK

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Abstract. A new method is proposed for the non-perturbative quantization of certain nonlinear field theories (group-bundle theories), based on a generalization of the idea of a promeasure from vector spaces to infinite-dimensional Lie groups. The quantum theory is not automatically finite, but there is a natural way of imposing a momentum cut-off, leading to the possibility of renormalization. The method relies on the geometrical structure of the classical theory and so may provide clues for the quantization of gravity.

1. Introduction

I shall propose a new method for quantizing certain nonlinear field theories having the following features (which are all thought to be highly desirable for quantum gravity).

(i) The quantization is linked to the geometrical structure of the fields and does not appeal to perturbation theory.

(ii) The method generalizes the Feynman integral and does not require Euclideanization (an important feature because in gravitation theory only a small number of metrics are capable of being Euclideanized).

(iii) The theory is exact (but, in consequence, pays the penalty of having to impose a momentum cut-off).

The quantization involves a fixed time-coordinate and so is not manifestly covariant. While this is a drawback, it does make possible a comparison with the usual Fock-space picture. Renormalization is an untackled problem; in non-perturbative terms, it amounts to finding a way of letting the momentum cut-off tend to infinity, while rescaling the Hilbert space, in such a way that the system tends to a limit.

Despite the motivation from quantum gravity, the approach here is not applicable as it stands to quantum gravity, but only to the simpler group-bundle theories introduced by Clarke (1979), which include skyrmions. While it is possible that the problem of quantum gravity will only finally yield to methods much more radical than the present one, it is hoped that the quantization of group-bundle theories might provide a 'testbed' for the general investigation of problems in quantizing nonlinear theories.

The organization of the paper is as follows. In section 2 I describe the classical theories to be quantized. The basis of the quantization method is to reverse the roles of the Schwarz space and its dual as used in the usual approaches (see, for example, Glimm and Jaffe 1972), and then generalize the underlying spaces to the infinite-dimensional Lie groups of fields arising from the theories of section 2. So in section

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3 I show how this reversal of spaces is possible for free fields, using as an example the free Klein–Gordon field, where everything is understood; then in section 4 I apply the reversed-space approach to group-bundle theories. Section 5 gives the conclusions.

2. Kink theories

These theories, described rigorously by Clarke (1979), are ones where the set of field values at a point forms a Lie group, with the group operation being independent of any choice of gauge. The collection of all these field values at all points forms, in geometrical language, a fibre bundle whose fibres are groups. It must be stressed that this is different from 'group space' approaches where the fibres are groups in any given gauge but where the group structure is gauge dependent; or where there is a group action but no group structure.

An example is the (3+1)-dimensional sine-Gordon equation over a fixed (curved) spacetime. The bundle is the set of tensors $\phi^{\mu}{}_{\nu}$ of rank (1, 1) satisfying

$$\phi^{\mu}{}_{\nu}\,\phi_{\mu}{}^{\sigma} = \delta_{\nu}{}^{\sigma} \tag{1}$$

the group operation is

$$(\phi, \psi) \mapsto \chi \qquad \chi^{\mu}{}_{\lambda} = \phi^{\mu}{}_{\nu}\,\psi^{\nu}{}_{\lambda} \tag{2}$$

and the field equation for sections ϕ of the bundle is

$$(\phi_{\mu\nu;\rho}\phi_{\lambda}{}^{\nu})_{;}{}^{\rho} - m^2\phi_{[\mu\lambda]} = 0 \tag{3}$$

(with the metric having signature $-+++$ and the usual raising/lowering conventions being in force). The name comes from the form taken by the static, spherically symmetric solutions in flat space when $\phi(x)$ is the matrix of a rotation about the radial direction through an angle θ satisfying

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} - \left(m^2 - \frac{2}{r^2}\right)\sin\theta = 0. \tag{4}$$

By analogy with the (1+1)-dimensional sine-Gordon equation, it is likely that this has particle-like solutions associated with 'topological charge' (i.e. solutions not homotopic to the trivial solutions in the set of finite-energy fields).

If we take the Lie algebra of the group formed by the field values at a point, and do this at every point, we get a related field theory whose bundle is a vector bundle. Linear equations of motion in this Lie-algebra theory correspond to nonlinear equations in the Lie-group-bundle theory, in a natural way. In the case of the example given, the fields in the Lie-algebra theory are skew-symmetric second-rank tensors with field equation

$$\psi_{\mu\nu;\rho}{}^{\rho} - m^2\psi_{\mu\nu} = 0. \tag{5}$$

The method of quantization depends on this relation between the linear algebra theory and the nonlinear group-bundle theory to lift functional integrals from one to the other.

3. Free-field integrals

I shall now reformulate simple free-field theory in a way which makes the transfer to a nonlinear theory possible. The basic tool is the idea of a cylinder-set measure, which replaces that of a measure.

A cylinder set Ω in a topological vector space V is a set such that there exists a closed vector subspace $Z \subset V$ with V/Z finite dimensional and a Borel set $\tilde{\Omega} \subset V/Z$ such that $\pi_Z^{-1}\tilde{\Omega} = \Omega$ (where $\pi_Z: V \to V/Z$ is a projection on cosets). A cylinder set measure μ on V is a function on the set of all cylinder sets of V such that for any fixed Z the 'restriction' μ_Z defined by

$$\mu_Z(\tilde{\Omega}) := \mu(\pi_Z^{-1}\tilde{\Omega}) \tag{6}$$

is a measure on V/Z (Gel'fand and Vilenkin 1964).

Using a cylinder set measure, one can define integrals of finitely based functions; that is, functions f of the form

$$f = f_Z \circ \pi_Z$$

for some closed Z with V/Z finite dimensional and some Lebesgue integrable

$$f_Z: V/Z \to \mathbb{C}.$$

We set

$$\int \mu(x) f(x) := \int \mu_Z(w) f_Z(w). \tag{7}$$

It is not hard to see that this is independent of the choice of Z.

So far all this generalizes to the case where V is an infinite-dimensional Lie group, as we shall show in the next section. But if V is a topological vector space, we have an alternative approach using the dual V'. For this, if $\delta_1, \ldots, \delta_n \in V'$ we can define

$$\pi_\sigma: V \to \mathbb{R}^n : x \mapsto ((x, \delta_1), \ldots, (x, \delta_n)) \tag{8}$$

(writing $\sigma = (\delta_1, \ldots, \delta_{n(\sigma)})$). If $\sigma^0 = \{x \in V | (\forall i)(x, \delta_i) = 0\}$, then the map $\theta_\sigma: \mathbb{R}^n \ni \xi \mapsto \pi_\sigma^{-1}(\xi) \in V/\sigma^0$ is a vector-space isomorphism, which can be used to express a cylinder set measure in terms of measures on $\mathbb{R}^{n(\sigma)}$ for varying σ. Thus a cylinder set measure can be defined alternately as a family of measures μ_σ on $\mathbb{R}^{n(\sigma)}$ with σ ranging over finite ordered subsets of V', satisfying appropriate consistency conditions, and a finitely based function f can be written as

$$f = \bar{f}_\sigma \circ \pi_\sigma \qquad f(x) = \bar{f}_\sigma((x, \delta_1), \ldots, (x, \delta_n)) \tag{9}$$

for some $\bar{f}_\sigma$. On this approach (Bourbaki 1969) a cylinder set measure is called a promeasure.

Given a positive-definite symmetric bilinear form q' on V', we can associate a promeasure with q' as follows. For a given $\sigma = (\delta_1, \ldots, \delta_n)$ let A be the matrix with entries $a_{ij} = q'(\delta_i, \delta_j)$, set $B = A^{-1}$ and for f given by (9) define

$$\int f \, d\mu_{q'} := (\det B/\pi^n)^{1/2} \int \bar{f}_\sigma(\xi) \exp[-b_{ij}\xi^i\xi^j)] \, d^n\xi \tag{10}$$

where b_{ij} are the entries of B. One can verify by direct calculation that this is independent of the choice of σ. If Ω is a cylinder set then $\mu_{q'}(\Omega)$ can be defined by taking f in (10) to be the characteristic function of Ω, thus making contact with our earlier definition (6).

The formalism can be extended to Feynman integrals by using Fresnel integrals (Albeverio and Hoegh-Krohn 1976) but we shall not go into this here.

We now apply this formalism to the real Klein-Gordon equation

$$\theta_{;\rho}{}^{\rho} - m^2\theta = 0 \tag{11}$$

of which (5) is a generalization. (In flat space the generalization is trivial, in that the components decouple, each satisfying (11).) As usual, we first impose periodic boundary conditions on θ at a given time t:

$$\theta(\boldsymbol{x}+L\boldsymbol{n})=\theta(\boldsymbol{x}) \tag{12}$$

for all $\boldsymbol{x}\in\mathbb{R}^3$ and $\boldsymbol{n}\in\mathbb{Z}^3$; the limit $L\to\infty$ will be taken later. For functions ψ, ϕ satisfying (12) we define an inner product

$$(\psi,\phi)_L:=\int_{V_L}\psi(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} \tag{13}$$

(where $V_L=\{(x,y,z)\,|\,0\leqslant x,y,z\leqslant L\}$) and we denote the real Hilbert space, formed by taking the closure of all such functions with this inner product, by H_L. This is the classical state-space.

We Fourier expand fields in terms of a Hilbert basis $\{\theta_{\boldsymbol{k}}^L\colon L\boldsymbol{k}\in\mathbb{Z}^3\}$, where we can take, for example,

$$\theta_{\boldsymbol{k}}^L(x):=\begin{cases}\eta_k\sin(2\pi\boldsymbol{k}\cdot\boldsymbol{x}) & k_1>0 \text{ or } k_1=0 \text{ and } k_2>0\\ & \text{or } k_1=k_2=0 \text{ and } k_3>0\\ \eta_k\cos(2\pi\boldsymbol{k}\cdot\boldsymbol{x}) & \text{otherwise}\end{cases}$$

where $\eta_k=\sqrt{8/L^3}(1/\sqrt{2})^{\#\{i:\,k_i=0\}}$. (The zero frequency mode will be excluded.) Quantization as a set of harmonic oscillators then proceeds as usual: the Fourier expansion

$$\theta(x)=\sum_{\boldsymbol{k}}c_{\boldsymbol{k}}\theta_{\boldsymbol{k}}^L(x) \tag{14}$$

defines a map

$$\rho:\theta\to(c_{\boldsymbol{k}})_{L\boldsymbol{k}\in\mathbb{Z}^3}\in l^2(\mathbb{Z}^3/L) \tag{15}$$

and with each coordinate $c_{\boldsymbol{k}}$ of θ we associate a wavefunction in $L^2(\mathbb{R},\mathbb{C})$ governed by a harmonic oscillator Hamiltonian with frequency $\omega_k=((2\pi\boldsymbol{k})^2+m^2)^{1/2}$. If $\psi_{n,\boldsymbol{k}}$ denotes the nth excited state of such a wavefunction, with

$$\psi_{0,\boldsymbol{k}}(c)=\left(\frac{\omega_k}{\hbar\pi}\right)^{1/4}\exp\{-\omega_k c^2/2\hbar\} \tag{16}$$

then the Fock-space state having $n_{\boldsymbol{k}}$ particles with momentum proportional to $\boldsymbol{k}$, and $n_{\boldsymbol{k}}=0$ for all but a finite number of $\boldsymbol{k}$, is the vector $\bigotimes_{\boldsymbol{k}}\psi_{n_{\boldsymbol{k}},\boldsymbol{k}}$ in the (non-separable) infinite tensor product $\bigotimes_{\boldsymbol{k}}L^2(\mathbb{R},\mathbb{C})$ (von Neumann 1939).

We can represent this as a function on state space H_L (or on $l^2(\mathbb{Z}^3/L)$, isomorphic to H_L via (15)) if we rescale the harmonic oscillator wavefunctions by maps $\alpha_{\boldsymbol{k}}$, where

$$\alpha_{\boldsymbol{k}}(\psi)(c):=\left(\frac{\hbar\pi}{\omega_k}\right)^{1/4}\mathrm{e}^{\omega_k c^2/2\hbar}\psi(c) \tag{17}$$

$$\alpha_{\boldsymbol{k}}:L^2(\mathbb{R},\mathbb{C})\to L^2(\mathbb{R},\mathbb{C},\mu_{\boldsymbol{k}})$$

with

$$\mathrm{d}\mu_{\boldsymbol{k}}(c)=\left(\frac{\omega_k}{\hbar\pi}\right)^{1/2}\mathrm{e}^{-\omega_k c^2/\hbar}\,\mathrm{d}c. \tag{18}$$

Then $\alpha_k(\psi_{0,k})=1$ and so we can multiply together all the $\alpha_k(\psi_{n_k,k})$ so as to define

$$\bar{\Psi}_\nu((c_k)_{k\in\mathbb{Z}^3/L}) := \prod_k \alpha_k(\psi_{n_k,k})(c_k). \tag{19}$$

$$\bar{\Psi}_\nu : l^2(\mathbb{Z}^3/L)\to\mathbb{C}$$

(still restricting to all but a finite number of n_k zero), with $\nu := (n_k)_{k\in\mathbb{Z}^3/L}$.

The wavefunction $\bar{\Psi}_\nu$ is a finitely based function since it depends on only a finite number of the c_k. To write the corresponding function on H_L, namely $\Psi_\nu=\bar{\Psi}_\nu\circ\rho$ (cf (15)), in the form of (9) we take the δ_i in the latter equation to be the set of θ_k^L for which $n_k\neq 0$. Then (19) is equivalent to

$$\Psi_\nu(\theta)=\bar{\Psi}_\nu(((\theta_k^L,\theta): n_k\neq 0)) \tag{20}$$

(where we now regard $\bar{\Psi}_\nu$ as a function of the finite set of those c_k for which $n_k\neq 0$), which is of the form (9).

The inner product in $L^2(\mathbb{R},\mathbb{C},\mu_k)$ gives rise to the bilinear form

$$(\Psi_\nu,\Psi'_\nu)=\prod_k\int \alpha_k(\psi_{n_k,k})(c_k)\overline{\alpha_k(\psi_{n'_k,k})}\,\mathrm{e}^{-\omega_k c_k^2/\hbar}\,\mathrm{d}c_k. \tag{21}$$

We can write this in the notation of (10) if we set $b_{kk'}:=\delta_{kk'}\omega_k/\hbar$, $a_{kk'}=\delta_{kk'}\hbar/\omega_k$ and $q'(\theta_k^L,\theta_{k'}^L):=a_{kk'}$. This defines q' as a bilinear form on H_L. Given Ψ_ν, $\Psi_{\nu'}$, regard $\bar{\Psi}_\nu$ and $\bar{\Psi}_{\nu'}$ as functions of all the c_k for which either $n_k\neq 0$ or $n'_k\neq 0$. Then using (20) and (10), (21) becomes

$$(\Psi_\nu,\Psi_{\nu'})=\int \mathrm{d}\mu_{q'}\Psi_\nu\bar{\Psi}_{\nu'}\,. \tag{22}$$

The completion of the linear span of the functions Ψ_ν with respect to this inner product (the quantum mechanical state space) will be denoted by $L^2(H_L,\mu_{q'})$.

To take the limit as $L\to\infty$, we note that if $L_1=mL_2$ for $m\in\mathbb{Z}$ then there is an inclusion map $\chi_{L_2L_1}: H_{L_2}\to H_{L_1}$. Thus the system of spaces H_L and maps $\chi_{L_2L_1}$ forms a direct system; we define H' to be its inductive limit (Dodson 1980, p 63). H' is dual to the space H of L^2 functions of compact support, and so we have a rigged Hilbert space of the form

$$H\subset L^2(\mathbb{R}^3)\subset H'. \tag{23}$$

q' can be defined by

$$q'(\phi,\psi)=(\phi,(m^2-\nabla^2)^{-1/2}\psi)$$

and the operator $(m^2-\nabla^2)^{-1/2}$ is well defined on H and can be extended to its maximal domain on H'. Thus in the limit we obtain a q' which defines a measure $\mu_{q'}$, not on all cylinder sets, but on those definable by using elements in the domain of q'. A similar restriction applies to finitely based functions. We continue to denote the space formed by completing the set of such functions by $L^2(H,\mu_{q'})$.

The dynamical evolution can be set up in terms of path integrals using these cylinder set measures. Briefly, if $\psi(c,t)$ is a harmonic oscillator wavefunction and $\tilde{\psi}:=\alpha(\psi)(c)$ (suppressing the frequency index $\boldsymbol{k}$ for the time being) then the Feynman path integral formula is

$$\tilde{\chi}(x,T)=\lim_{n\to\infty}\frac{\int\ldots\int\tilde{\psi}(c_0,0)\exp[-(\mathrm{i}/\hbar)S_0(\lambda_0(c_0,\ldots,c_{n-1},c))]\,\mathrm{d}\mu(c_0)\ldots\mathrm{d}\mu(c_{n-1})}{\int\ldots\int\exp[-(\mathrm{i}/\hbar)S_0(\lambda_n(c_0,\ldots,c_{n-1},c))]\,\mathrm{d}\mu(c_0)\ldots\mathrm{d}\mu(c_{n-1})} \tag{24}$$

(Feynman and Hibbs 1965), where

$$S_0(\gamma) := \int_0^T \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)^2 \mathrm{d}t$$

$$\lambda_n : \mathbb{R}^{n+1} \to c([0, T], \mathbb{R})$$

$$\lambda_n(c)(t) = c_i(nt - i + 1) + c_{i-1}(i - nt) \qquad t \in \left[\frac{i-1}{n}, \frac{i}{n}\right].$$

The measures $\mathrm{d}\mu$ (given by (18)) both ensure the convergence of the integrals and allow us to use a purely kinetic action S_0. On passing to Ψ_ν we obtain a precisely analogous expression to (24) with $\mathrm{d}\mu$ replaced by $\mathrm{d}\mu_{q'}$, the c's replaced by fields θ_i and S_0 replaced by

$$S(\gamma) := \int_0^T \mathrm{d}^3x\, \dot{\gamma}(x, t)^2\, \mathrm{d}t.$$

4. Group bundle integrals

In the previous section we obtained a formulation of free-field quantum theory using finitely based functions on the space H of L^2 functions on $\mathbb{R}^3$ of compact support (23). It is important to note that the roles of the space and its dual are the reverse of the usual representation (Glimm and Jaffe 1972) where one uses, for example, the rigging

$$\zeta' \supset L^2(\mathbb{R}^3) \supset \zeta \tag{25}$$

where ζ is the Schwarz space of rapidly decreasing functions and ζ' the corresponding space of distributions, with ζ' being regarded as the configuration space. The point is that if restricted finitely based functions are used, followed by a formal completion, it is immaterial what space is used as the state space.

We now extend the idea to group-bundle theories, stressing again that by this we mean theories with a gauge-independent group operation on the fields at each point. For simplicity of description, suppose the background geometry is Minkowskian. Then the set of all fields at a given time t_0 (i.e. all sections of the restriction of the bundle to the hypersurface $t = t_0$) forms a group G with the group operation being pointwise multiplication

$$(\phi\psi)(x) = \phi(x)\psi(x). \tag{26}$$

If we specify a topology on the fields, the group becomes a Lie group, and its Lie algebra is the set of fields in the related linear Lie-algebra theory, which we suppose quantized by a generalization of the approach in the previous section.

To perform the same quantization in G we need to define cylinder sets in G. The definition is obvious: a cylinder set is a set Ω such that there exists a closed Lie subgroup $\kappa \subset G$ with G/κ finite dimensional and a Borel set $\tilde{\Omega} \subset G/\kappa$ such that $\pi_\kappa^{-1}\tilde{\Omega} = \Omega$, where $\pi_\kappa : \to G/\kappa$ is projection on cosets. Cylinder set measures and finitely based functions can then be defined as in (6), (7).

It is here that the 'divergences' of quantum field theory have to be faced. The obvious approach is to base the topology of G on that of the state space in the linear theory, the space H of L^2 functions of compact support. We could do this by using the exponential map

$$\exp : H \to G \tag{27}$$

defining G to be the image of H with the induced topology. But we have to remember that, strictly speaking, H consists of functions, modulo functions that are zero except on sets of measure zero; so that G is defined as a set of equivalence classes of fields differing on sets of measure zero. But in this case the only subgroup κ with G/κ a finite-dimensional Lie group is G itself, with G/κ trivial! If we had used the group of continuous fields, instead of 'L^2 fields', then every κ with the required properties would have the form

$$\kappa\{\phi: (\exists n)(p_1, \ldots, p_n)(\phi(p_1) = e_{p_1}, \ldots, \phi(p_n) = e_{p_n})\} \tag{28}$$

where e_x is the identity in the fibre at x. But when we take equivalence classes, the values of fields at a finite set of points have no significance and κ is identified with the whole of G.

In order to allow such a κ, distinct from G, we must use continuous fields. Then in the linear theory, the cylinder sets corresponding to κ are defined in terms of the elements $\delta_i^{(3)} \in V'$, where

$$(\phi, \delta_i^{(3)}) = \phi(p_i).$$

Explicitly, a finitely based function using these has the form (from (9))

$$f(\phi) = \bar{f}(\phi(p_1), \ldots, \phi(p_n)). \tag{29}$$

Thus q' has to be defineable on elements like $\delta_i^{(3)}$. But in fact q' does not extend so that $q'(\delta_i^{(3)}, \delta_i^{(3)})$ is defined, unless we impose a cut-off in the momentum. So q' is modified accordingly.

We now have the problem of transferring the cylinder set measure defined by (10) in the Lie algebra theory into the group-bundle theory; i.e. of generalizing a Gaussian distribution to a non-Abelian Lie group. The obvious candidate for this is a heat-equation kernel. In view of the close relation between the heat equation and the Wiener process, it is likely that this choice is equivalent to the method of transferring path integrals to non-flat manifolds described by Elworthy (1983).

To define this, let β be the bilinear form on the Lie algebra to G/κ having components b_{ij} with respect to a basis $(\alpha_i)_{i=1}^n$ dual to the basis $(\delta_i)_{i=1}^n$ used in (10) (where b_{ij} is defined). Regarding β as a Riemannian metric on G/κ we can define the usual Hodge *-operation, derivative $\delta = {}^*d^*$ and Laplacian $\Delta = d\delta + \delta d$. Finally define an n-form $\rho_\kappa(\cdot, \tau)$ depending on a parameter τ by the heat equation

$$\frac{\partial}{\partial \tau} \rho_\kappa = \Delta \rho_\kappa \tag{30a}$$

$$\rho_\kappa(\cdot, \tau) \to \delta_e \qquad (\tau \to 0) \tag{30b}$$

where δ_e is the unit point measure at the identity. We regard ρ_κ as a measure and, for a cylinder set $\Omega = \pi_\kappa^{-1}\tilde{\Omega}$, we define

$$\mu_{q'}(\Omega) = \int_{\tilde{\Omega}} \rho_\kappa(\cdot, 1). \tag{31}$$

One can then show by some elementary differential geometry that this is independent of the choice of κ.

Once we have a cylinder set measure we can proceed exactly as before, defining $L^2(G, \mu_{q'})$ for the quantum Hilbert space. For the evolution we can write down essentially the same Feynman path integral as before with λ_n defined by interpolation using exponentiation in the relevant Lie groups. Some work remains to be done here to show that the resulting function is in $L^2(G, \mu_{q'})$ (i.e. that it can be approximated in the L^2 sense by finitely based functions). In the linear theory the corresponding function was already finitely based, but this concept has now become much narrower. But there is no reason to think that this is more than a matter of technical verification.

5. Conclusion

It will be seen that, while our motivation has been drawn from 'kink theories', the construction does not depend on the gauge-invariance of the group operation: all that is needed is the gauge-invariance of the defined promeasures. Moreover, the group operation only actually enters as a means of defining a Riemannian metric on certain quotient groups, and one would expect this to be possible in much more general contexts. Thus the indications are that the method can be used to construct exact momentum cut-off models for a wide range of nonlinear theories in a completely non-perturbative way.

It is obvious that renormalization (in the sense described in the introduction) is the major obstacle still to be tackled. But the prospects for the method seem at present very good.

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